Example 5.5 (Beam Equation)**.** The Beam Equation provides a model for the load carrying and deflection properties of beams, and is given by

$$\frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial^4 u}{\partial x^4} = 0.$$

... but you won't see them in this course. You'll have to wait until Maths for Engineers 3 (MATH6503) for that!

5.2 First order separable ODEs

An ODE $\frac{\mathrm{d}y}{\mathrm{d}x} = F(x, y)$ is *separable* if we can write $F(x, y) = f(x)g(y)$ for some functions $f(x)$, $g(y)$.

Example 5.6.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y \qquad \text{IS separable,}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 - y^2 \qquad \text{IS NOT.}$$

Example 5.7. Find the general solution to the ODE

$$9y\frac{\mathrm{d}y}{\mathrm{d}x} + 4x = 0.$$

"Separating the variables", we have

$$\begin{aligned} 9y\mathrm{d}y &= -4x\mathrm{d}x \iff \\ 9\int y\mathrm{d}y &= -4\int x\mathrm{d}x \\ \frac{9}{2}y^2 &= -\frac{4}{2}x^2 + C, \end{aligned}$$

i.e. the general solution is

$$\frac{x^2}{9} + \frac{y^2}{4} = K, \quad (K = C/36)$$

which describes a 'family' of ellipses.

We can check our solution by differentiating:

$$\frac{2}{9}x + \frac{2}{4}yy' = 0$$

i.e

$$9yy' + 4x = 0.$$

Example 5.8. Find the general solution to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y+1}{x+1}.$$

$$\begin{aligned}
&\Rightarrow \int \frac{1}{y+1}\mathrm{d}y = \int \frac{1}{x+1}\mathrm{d}x \\
&\Rightarrow \quad \ln|y+1| = \ln|x+1| + C.
\end{aligned}$$

Use $\log\left(\frac{a}{b}\right) = \log a - \log b$:

$$\ln\left|\frac{y+1}{x+1}\right| = C,$$

or

$$\frac{y+1}{x+1} = e^C = K.$$

Again we can easily check this using differentiation.

Example 5.9. Solve the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y^2$$

Separating variables:

$$\begin{aligned}
\int \frac{\mathrm{d}y}{1+y^2} &= \int \mathrm{d}x \\
\Rightarrow \quad \arctan y &= x + C \\
\Rightarrow \quad y &= \tan(x+C).
\end{aligned}$$

Once again, this is easily checked by differentiation.

Example 5.10 (2007 Exam Question)**.** Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y(y+1)}{x(x-1)} = 0$$

finding y explicitly, i.e $y = f(x)$.

Solution: This equation is separable, thus separating the variables and integrating gives

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y(y+1)}{x(x-1)} \\
\int \frac{\mathrm{d}y}{y(y+1)} &= \int \frac{\mathrm{d}x}{x(x-1)}.
\end{aligned}$$

To solve the integrals, use partial fractions:

$$\begin{aligned}
\int \left[\frac{1}{y} - \frac{1}{y+1}\right]\mathrm{d}y &= \int \left[-\frac{1}{x} + \frac{1}{x-1}\right]\mathrm{d}x \\
\ln y - \ln(y+1) &= -\ln x + \ln(x-1) + C \\
\ln\left(\frac{y}{y+1}\right) &= \ln\left(\frac{x-1}{x}\right) + C \\
\frac{y+1}{y} &= e^{-C}\frac{x}{x-1}.
\end{aligned}$$

Let $K = e^C$. Then

$$\begin{aligned}
y &= (y+1)\left(\frac{x-1}{Kx}\right) \\
y\left[1 - \left(\frac{x-1}{Kx}\right)\right] &= \left(\frac{x-1}{Kx}\right) \\
y(Kx - x + 1) &= x - 1.
\end{aligned}$$

$$\therefore \quad y = \frac{x-1}{Kx - x + 1}$$

is the explicit solution.

Example 5.11 (2010 Exam Question)**.** Solve

$$(y + x^2 y)\frac{\mathrm{d}y}{\mathrm{d}x} = 1.$$

Solution:

$$\begin{aligned} y(1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x} &= 1 \\ \int y\,\mathrm{d}y &= \int \frac{\mathrm{d}x}{x^2+1} \\ \frac{y^2}{2} &= \arctan x + C \end{aligned}$$

i.e. the solution is $y = \pm\sqrt{2\arctan x + 2C}$.

5.3 First order linear ODEs

Aside: Exact types An *exact type* is where the LHS of the differential equation is the exact derivative of the product.

Example 5.12.

$$\begin{aligned} & x\frac{\mathrm{d}y}{\mathrm{d}x} + y = e^x \\ \Rightarrow \quad & \frac{\mathrm{d}}{\mathrm{d}x}(xy) = e^x \\ \Rightarrow \quad & xy = e^x + C. \end{aligned}$$

Example 5.13.

$$\begin{aligned} & e^x e^y \frac{\mathrm{d}y}{\mathrm{d}x} + e^x e^y = e^{2x} \\ \Rightarrow \quad & \frac{\mathrm{d}}{\mathrm{d}x}(e^x e^y) = e^{2x} \\ \Rightarrow \quad & e^x e^y = \frac{1}{2}e^{2x} + C. \end{aligned}$$

I recommend that you bear this in mind as we proceed...

First order linear ODEs are equations that may be written in the form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x). \tag{5.2}$$

Example 5.14.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\cot x = \operatorname{cosec} x. \qquad [P(x) = \cot x,\ Q(x) = \operatorname{cosec} x]$$

Example 5.15.

$$\begin{aligned}
&\tan x \frac{\mathrm{d}y}{\mathrm{d}x} + y = e^x \tan x \\
\Rightarrow\quad &\frac{\mathrm{d}y}{\mathrm{d}x} + \cot x\, y = e^x. \quad [P(x) = \cot x,\ Q(x) = e^x]
\end{aligned}$$

In general, Equation (5.2) is NOT exact.

Big question: Can we multiply the equation by a function of x which will make it exact?

Let's suppose we can, and call this function $I(x)$; the *integrating factor (IF)*. Then multiply both sides of (5.2) by I:

$$\underbrace{I\frac{\mathrm{d}y}{\mathrm{d}x} + IPy}_{\text{Exact type}} = IQ.$$

Compare the LHS with

$$\overbrace{I\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}I}{\mathrm{d}x}y}^{\frac{\mathrm{d}}{\mathrm{d}x}(Iy)},$$

Hence we require

$$\begin{aligned}
&IP\cancel{y} = \frac{\mathrm{d}I}{\mathrm{d}x}\cancel{y} \\
\Rightarrow\quad &\frac{\mathrm{d}I}{\mathrm{d}x} = IP \\
\Rightarrow\quad &\int \frac{\mathrm{d}I}{I} = \int P\,\mathrm{d}x \\
\Rightarrow\quad &\ln I = \int P\,\mathrm{d}x \qquad \text{[No need for integration constants!]} \\
\Rightarrow\quad &\ln I = e^{\int P\,\mathrm{d}x},
\end{aligned}$$

and this is the IF. We will substitute this into (5.2):

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x).$$

Multiply by I:

$$\begin{aligned}
&e^{\int P\,\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x} + e^{\int P\,\mathrm{d}x}Py = e^{\int P\,\mathrm{d}x}Q \\
\Rightarrow\quad &\frac{\mathrm{d}}{\mathrm{d}x}(ye^{\int P\,\mathrm{d}x}) = e^{\int P\,\mathrm{d}x}Q \\
\Rightarrow\quad &yI = \int e^{\int P\,\mathrm{d}x}Q\,\mathrm{d}x.
\end{aligned}$$

This is the form we end up with.

I will not ask you to go through this derivation in the exam. However, you will need to know how to apply it.

Example 5.16. Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = e^{-x}.$$

We require the IF:

$$I = e^{\int P\,\mathrm{d}x} = e^{\int 2\,\mathrm{d}x} = e^{2x}.$$

Then

$$\begin{aligned} & e^{2x}\frac{\mathrm{d}y}{\mathrm{d}x} + 2e^{2x}y = e^{2x}e^{-x} \\ \Rightarrow\quad & \frac{\mathrm{d}}{\mathrm{d}x}(ye^{2x}) = e^{x} \\ \Rightarrow\quad & ye^{2x} = e^{x} + C, \end{aligned}$$

or

$$y = e^{-x} + Ce^{-2x}.$$

Example 5.17. Solve

$$\cos x\frac{\mathrm{d}y}{\mathrm{d}x} + y\sin x = \frac{1}{2}\sin 2x.$$

Get it into the right form first!

$$\begin{aligned} \Rightarrow\quad & \frac{\mathrm{d}y}{\mathrm{d}x} + y\tan x = \frac{\sin 2x}{2\cos x} = \frac{\cancel{2}\sin x\cancel{\cos x}}{\cancel{2}\cancel{\cos x}} \\ \Rightarrow\quad & \frac{\mathrm{d}y}{\mathrm{d}x} + y\tan x = \sin x, \end{aligned} \tag{5.3}$$

so $P(x) = \tan x$. Now seek the IF:

$$I = e^{\int P\,\mathrm{d}x} = e^{\int \tan x\,\mathrm{d}x} = e^{-\ln(\cos x)} = \frac{1}{e^{\ln(\cos x)}} = \frac{1}{\cos x}.$$

A VERY common error: $e^{-\ln(\cos x)} = \cos x$.

Multiply (5.3) throughout by I to give

$$\frac{1}{\cos x}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\tan x}{\cos x}y = \tan x,$$

i.e.

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\cos x}\right) = \tan x \\ \Rightarrow\quad & \frac{y}{\cos x} = \int \tan x\,\mathrm{d}x + C = -\ln(\cos x) + C. \end{aligned}$$

Therefore the general solution is

$$y = C\cos x - \cos x\ln(\cos x).$$

Example 5.18. Solve

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + = x^2 + 3y.$$

Get it in the right form first...

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{3}{x}y = x. \tag{5.4}$$

Find the integrating factor

$$I(x) = e^{\int -\frac{3}{x}\,\mathrm{d}x} = e^{-3\ln x} = e^{\ln(x^{-3})} = x^{-3},$$

Now multiply both sides of (5.4) by the integrating factor to make the LHS an exact type:

$$x^{-3}\frac{\mathrm{d}y}{\mathrm{d}x} - 3x^{-4}y = x^{-2}\frac{\partial}{\partial x}\left(x^{-3}y\right) \qquad = x^{-2},$$

and integrate both sides of the equation to gain

$$\begin{aligned} x^{-3}y &= -x^{-1} + C \\ y &= x^3\left(C - x^{-1}\right) \\ y &= x^2(Cx - 1). \end{aligned}$$

5.4 Initial Value Problems

All the solutions we obtained so far contain an annoying constant of integration C. When engineers work with ODEs, they are interested in a particular solution satisfying the given *initial condition*.

An ODE together with an initial condition (IC) is called an *initial value problem (IVP)*. In other words:

$$\text{ODE} + \text{IC} = \text{IVP}$$

We need only two steps to solve an IVP:

1 ODE: Find the general solution, containing an arbitrary constant.

2 IC: Apply the condition to determine the arbitrary constant. Usually, the condition is given as

$$y(x_0) = y_0,$$

which tells us that when $x = x_0$, $y = y_0$.

Example 5.19. Solve the IVP

$$2\frac{\mathrm{d}y}{\mathrm{d}x} - 4xy = 2x, \quad y(0) = 0.$$

Start by rewriting in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 2xy = x,$$

which is a first order linear equation, so we calculate the IF:

$$I = e^{\int -2x\,\mathrm{d}x} = e^{-x^2}.$$

$$\therefore \quad \frac{\mathrm{d}y}{\mathrm{d}x}e^{-x^2} - 2xe^{-x^2}y = xe^{-x^2}.$$

Hence

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left(ye^{-x^2}\right) = xe^{-x^2} \\
\Rightarrow\quad & ye^{-x^2} = \int xe^{-x^2}\,\mathrm{d}x, \\
\Rightarrow\quad & ye^{-x^2} = -\frac{1}{2}e^{-x^2} + C \\
\Rightarrow\quad & y = -\frac{1}{2} + Ce^{x^2}.
\end{aligned}$$

Now apply the IC $y(0) = 0$. This gives

$$0 = -\frac{1}{2} + C \quad \Rightarrow C = \frac{1}{2},$$

and so the solution is

$$y = \frac{1}{2}\left(e^{x^2} - 1\right).$$

Example 5.20. Solve the IVP

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 4x^2, \quad y(1) = 2.$$

Get the equation in the right form first!

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x}y = 4x.$$

Then the IF is:

$$\begin{aligned}
& I = \mathrm{e}^{\int \frac{2}{x}\mathrm{d}x} = e^{2\ln x} = e^{\ln x^2} = x^2. \\
\Rightarrow\quad & x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 4x^3 \\
\Rightarrow\quad & \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2y\right) = 4x^3 \\
\Rightarrow\quad & x^2y = x^4 + C \\
\Rightarrow\quad & y = x^2 + Cx^{-2}.
\end{aligned}$$

Apply the condition $y(1) = 2$:

$$y(1) = 1 + C = 2 \quad \Rightarrow \quad C = 1.$$

So the solution is

$$y = x^2 + \frac{1}{x^2}.$$

Example 5.21 (Logistic Equation)**.** Suppose the rate of change of x is proportional to:

$$rx\,(1 - x)\,,$$

where $r > 0$ is constant. Show that if initially $x = x_0$ (at $t = 0$) and $0 < x_0 < 1$, then $\lim_{t\to\infty} x = 1$.

First, we set up the ODE:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\,(1 - x)\,,$$

which is the logistic equation. This ODE has applications in many fields of study such as ecology, psychology, chemistry and even politics!

The logistic equation can be tackled by separating variables...

$$
\begin{aligned}
\int \frac{\mathrm{d}x}{x\,(1-x)} &= r\int \mathrm{d}t \\
\int \left[\frac{1}{x}+\frac{1}{1-x}\right]\mathrm{d}x &= rt + C \\
\ln|x| - \ln|1-x| &= rt + C \\
\ln|\frac{x}{1-x}| &= rt + C \\
\frac{x}{1-x} &= e^{rt+C} = e^{rt}e^{C},
\end{aligned}
$$

and let $G = e^C$. We then make x the subject...

$$
\begin{aligned}
x &= (1-x)Ge^{rt} \\
x &= Ge^{rt} - xGe^{rt} \\
x(1+Ge^{rt}) &= Ge^{rt},
\end{aligned}
$$

which leads to

$$x = \frac{Ge^{rt}}{1+Ge^{rt}}.$$

Next, find G using the initial condition:

$$x_0 = \frac{1}{\frac{1}{G}+1}, \quad \Rightarrow \quad \frac{1}{G} = \frac{1}{x_0} - 1,$$

and therefore

$$x(t) = \frac{1}{1+\left(\frac{1}{x_0}-1\right)}e^{-rt} = \frac{x_0}{x_0+(1-x_0)e^{-rt}},$$

the so-called logistic function. Finally, we note that as $t \to \infty$, $x(t) \to \frac{\cancel{x_0}}{\cancel{x_0}} = 1$, as intended.

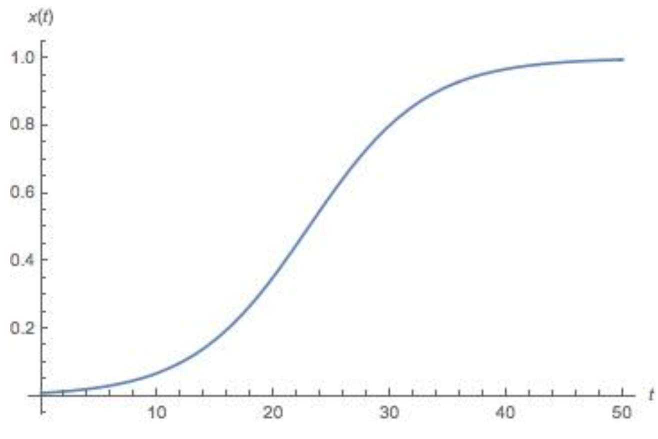

Figure 5.3: A plot depicting the logistic curve. Here, $x_0 = 0.01$ and $r = 0.2$.